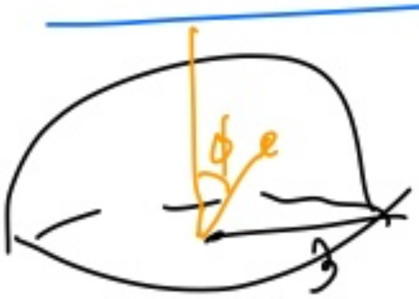
(3.8) Avg value of z in upper half of ball of radius 3 centered at 0.



$$\begin{pmatrix}\text{Avg value}\\ \text{of } f(x,y,z)\end{pmatrix} = \frac{\displaystyle\iiint_R \overset{z}{f(x,y,z)}\,dx\,dy\,dz}{\displaystyle\iiint_R 1\,dx\,dy\,dz}$$

Volume of region

$$\frac{1}{2}\,\frac{4}{3}\pi(3)^3 = 18\pi$$

Spherical:

$0 \le \rho \le 3$

$0 \le \phi \le \frac{\pi}{2}$

$0 \le \theta \le 2\pi$

$z = \rho\cos\phi$

$dx\,dy\,dz$

$= \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

Top integral $= \displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^3 (\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$\frac{\rho^4}{4}\Big|_0^3 \leftarrow \frac{81}{4}$

$$= \underset{2\pi}{\cancel{\int_0^{2\pi}}} \left(\int_0^{\pi/2} \frac{81}{4} \cos\phi \sin\phi \, d\phi \right) \cancel{d\theta}$$

$$u = \sin\phi \qquad 0 \le u \le 1$$

$$du = \cos\phi \, d\phi$$

$$= 2\pi \left(\frac{81}{4}\right) \int_0^1 u \, du$$

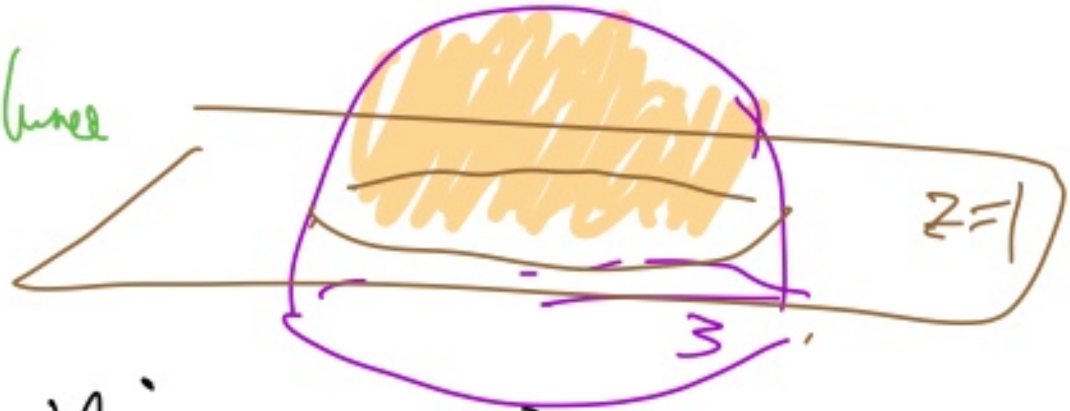
$$\left.\frac{u^2}{2}\right|_0^1 \to \frac{1}{2}$$

$$= \frac{2\pi\, 81}{4} \left(\frac{1}{2}\right) = \frac{81\pi}{4} = \text{top}$$

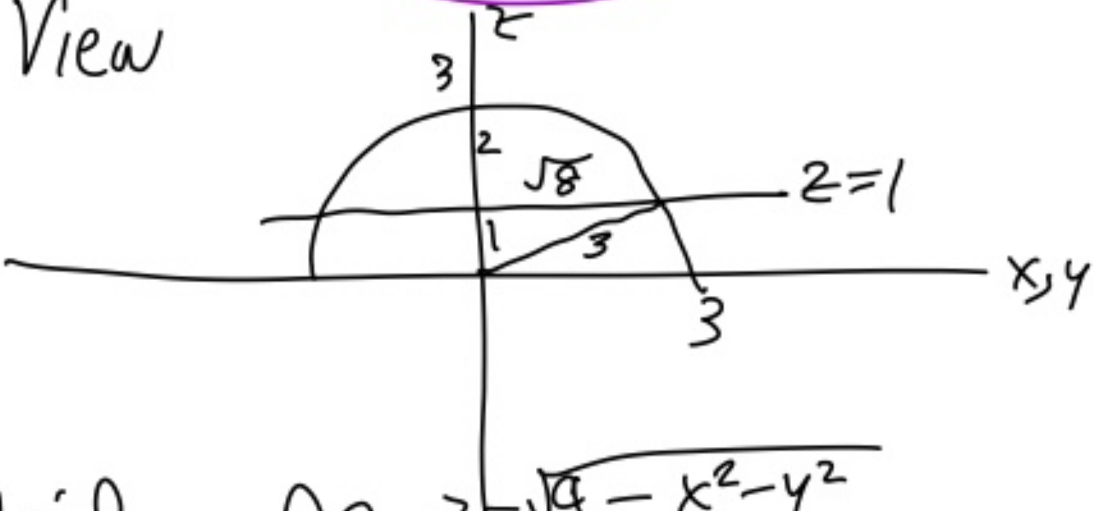
$$\Rightarrow \left(\begin{array}{c}\text{avg value of} \\ z\end{array}\right) = \frac{81\pi/4}{18\pi} = \frac{\cancel{3^4}^{\,9}\,\cancel{\pi}}{\underset{8}{\cancel{2^2}} \cdot \cancel{3^2} \cdot \cancel{2\pi}}$$

$$= \boxed{\frac{9}{8}}.$$

3.12 Volume



Side View



(b) Cylindrical.
$$\iint_{xy} \int_{z=1}^{z=\sqrt{9-x^2-y^2}} 1 \, dz \, dx\,dy$$

polar

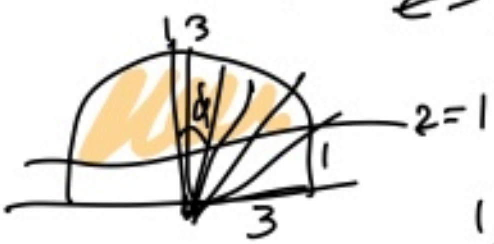
$x^2+y^2 \le 8$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{8}} \int_{z=1}^{\sqrt{9-r^2}} 1 \, dz \, r\,dr\,d\theta$$

$$\int_0^{2\pi} \left(\int_0^{\sqrt{8}} \left(r\sqrt{9-r^2} - r \right) dr \right) d\theta$$

(a) Spherical

$z=1 \quad \rho\cos\phi = 1 \Rightarrow \rho = \frac{1}{\cos\phi}$ bottom.

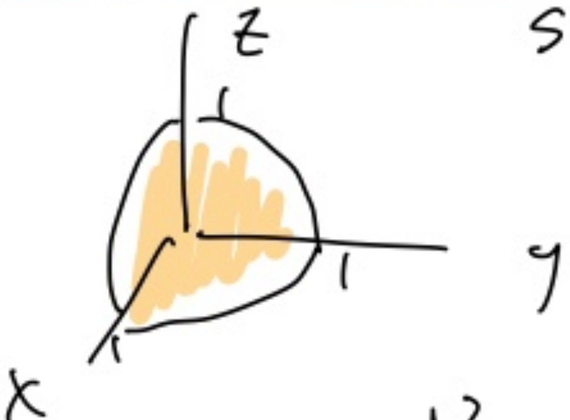


$\frac{1}{\cos\phi} \le \rho \le 3$

$0 \le \theta \le 2\pi$

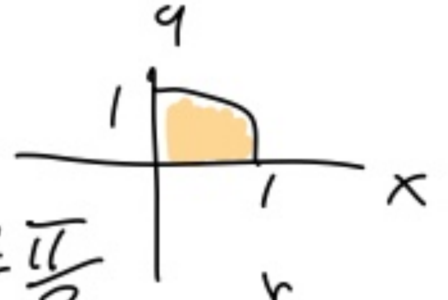
$0 \le \phi \le \arccos\left(\frac{1}{3}\right)$

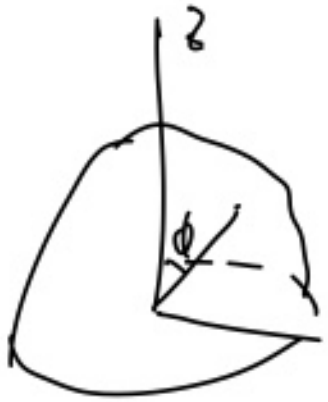
Find the average value of y^2 on the part of the unit ball in the first octant.



spherical $0 \le \rho \le 1$

$0 \le \theta \le \frac{\pi}{2}$





$0 \le \phi \le \frac{\pi}{2}$

$y = \overbrace{\rho \sin\phi}^{r} \sin\theta$

$y^2 = \rho^2 \sin^2\phi \sin^2\theta$

$$\iiint_{\text{region}} y^2 \, dz\, dy\, dx$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{1} \rho^2 \sin^2\phi \sin^2\theta \left(\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta\right)$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \underbrace{\int_0^1 \rho^4 \, d\rho}_{\frac{\rho^5}{5}\Big|_0^1} \sin^3\phi \, d\phi \, \sin^2\theta \, d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{5}\sin^3\phi \, d\phi \, \sin^2\theta \, d\theta$$

$(\sin^2\phi)\sin\phi$

$(1-\cos^2\phi)$

$1 \le u \le 0$

$u = \cos\phi$

$du = -\sin\phi \, d\phi$

$$= \frac{1}{5}\int_0^{\pi/2}\int_1^{0} -(1-u^2)\,du \, \sin^2\theta \, d\theta$$

$$= \frac{1}{5}\int_0^{\pi/2}\int_0^{1}(1-u^2)\,du \, \sin^2\theta \, d\theta$$

$u - \frac{u^3}{3}\Big|_0^1$

$1 - \frac{1}{3} = \frac{2}{3}$

$$= \frac{2}{15}\int_0^{\pi/2} \sin^2\theta \, d\theta$$

$\frac{1}{2} - \frac{1}{2}\cos(2\theta)$

$\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\Big|_0^{\pi/2}$

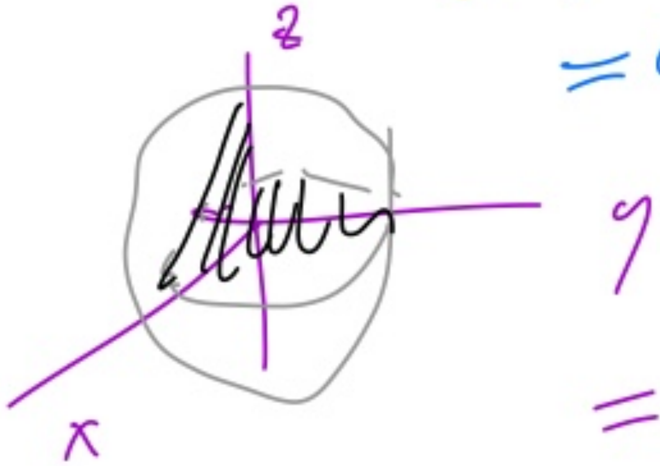
$$= \frac{2}{15}\frac{\pi}{4} = \frac{\pi}{30}$$ fop $\frac{\pi}{4}$

$$\iiint y^2 = \frac{\pi}{30}$$

$$\text{bottom} = \iiint 1 = \text{volume} = \frac{4}{3}\pi(1)^3 \cdot \frac{1}{8} = \frac{\pi}{6}$$

$$\text{avg value of } y^2 = \frac{\pi/30}{\pi/6} = \frac{\cancel{\pi}}{\cancel{30}_5} \cdot \frac{\cancel{6}^1}{\cancel{\pi}} = \boxed{\frac{1}{5}}.$$

Sneaky way to find the average value of y^2 over 1^{st} octant:



= avg value of y^2 on the whole unit ball.

= avg value of x^2 on whole unit ball

= avg value of z^2 on whole unit ball.

$= \frac{1}{3}$ avg value of $x^2+y^2+z^2 = \rho^2$ on unit ball.

$$= \frac{1}{3} \cdot \frac{\iiint_{\text{ball}} \rho^2}{\iiint_{\text{ball}} 1}$$

$$= \frac{1}{3} \frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}{\underbrace{\text{volume(unit ball)}}_{\frac{4}{3}\pi}}$$

$$\frac{\frac{1}{\cancel{3}} \int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5}\Big|_0^1 \sin\phi \, d\phi \, d\theta}{\frac{4\pi}{\cancel{3}}}$$

$$\frac{1}{5} \cdot \frac{1}{4\pi} \underbrace{\int_0^{2\pi} \underbrace{\underbrace{-\cos\phi \Big|_0^{\pi}}_{-(-1)-(-1)}}_{2} \, d\theta}_{4\pi}$$

$$= \boxed{\frac{1}{5}}$$

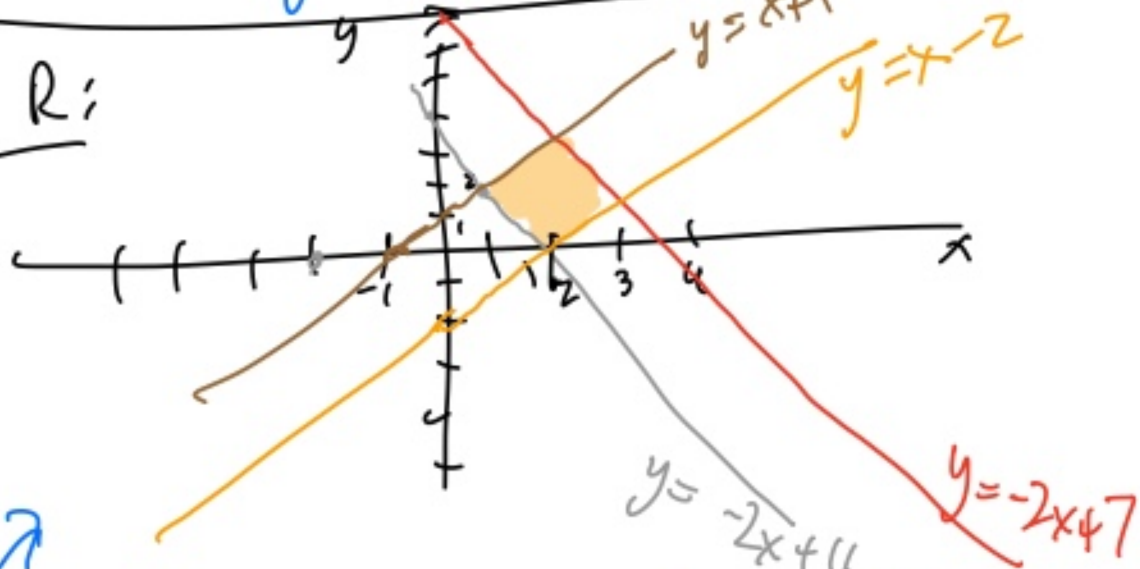
Other changes of coordinates:

Example Find $\iint_R (2x^2 - xy - y^2)\,dx\,dy$,

where R is the region bounded by

$$y = -2x+4$$
$$y = -2x+7$$
$$y = x-2$$
$$y = x+1$$

Graph R:



integral in these coordinates will be tough

Let $u = x - y \Rightarrow -1 = x - y = u$

$2 = x - y = u$

Let $v = 2x + y$

$7 = 2x + y = v$

$4 = 2x + y = v$

$-1 \le u \le 2$
$4 \le v \le 7$

Vadd

$$u+v=3x \Rightarrow x=\frac{u+v}{3}$$

$$u=x-y \Rightarrow y=x-u$$

$$=\frac{u+v}{3}-u\frac{3}{3}$$

$$y=\frac{-2u+v}{3}$$

$$=-\frac{2}{3}u+\frac{1}{3}v$$

$$dx=\frac{1}{3}du+\frac{1}{3}dv$$

$$dy=\frac{-2}{3}du+\frac{1}{3}dv$$

$$dx\wedge dy=\left(\frac{1}{3}du+\frac{1}{3}dv\right)\wedge\left(\frac{-2}{3}du+\frac{1}{3}dv\right)$$

$$\frac{1}{3}du\wedge\frac{1}{3}dv+\frac{1}{3}dv\wedge\left(-\frac{2}{3}du\right)$$

$$=\left(\frac{1}{9}+\frac{2}{9}\right)du\wedge dv$$

$$=\frac{1}{3}du\wedge dv$$

function: $2x^2 - xy - y^2$

$$= \frac{2(u+v)^2}{9} - \frac{(u+v)(-2u+v)}{9} - \frac{(-2u+v)^2}{9}$$

$$= \frac{1}{9}\Big[2(\cancel{u^2}+2uv+v^2) - (\cancel{-2u^2}-uv+v^2) - (\cancel{4u^2}-4uv+v^2)\Big]$$

$$= \frac{1}{9}\big[9uv\big] = uv$$

Our integral is

$$\int_4^7 \int_{-1}^{2} uv \, \frac{1}{3} \, du \, dv$$

$$= \frac{1}{3}\int_4^7 \frac{u^2}{2}\Big|_{-1}^{2} v \, dv$$

$$\frac{4-1}{2} = \frac{3}{2}$$

$$= \frac{1}{2}\int_4^7 v \, dv = \frac{1}{2}\,\frac{v^2}{2}\Big|_4^7$$

$$\frac{49-16}{2}$$

$$= \boxed{\frac{33}{4}}$$

$\boxed{\text{Example}}$ Consider $\int_1^2 \int_1^2 \frac{y}{x}\,dy\,dx$.

Change variables using

$$x = u$$
$$y = uv$$

---

function: $\frac{y}{x} = \frac{uv}{u} = v$

volume form $dx \wedge dy = du \wedge (v\,du + u\,dv)$

$$= u\,du \wedge dv$$

$$\iint u\cancel{v}\,du\,dv$$

$$1 \le x \le 2 \Rightarrow 1 \le u \le 2$$

$1 \le y \le 2 \Rightarrow 1 \le uv \le 2$

$$\frac{1}{u} \le v \le \frac{2}{u}$$

our integral is

$$\int_{u=1}^{2} \int_{v=\frac{1}{u}}^{\frac{2}{u}} uv \, dv \, du$$

$$= \int_1^2 \frac{uv^2}{2} \Big|_{v=1/u}^{v=2/u} du$$

$$\frac{u}{2}\left(\frac{4}{u^2} - \frac{1}{u^2}\right)$$

$$= \int_1^2 \frac{3}{2u} \, du = \frac{3}{2} \ln|u| \Big|_1^2$$

$$= \frac{3}{2}(\ln 2 - \ln(1))$$

$$= \boxed{\frac{3}{2} \ln 2}$$

Original integral

$$\int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx$$

$$= \int_1^2 \frac{1}{x}\left(\frac{y^2}{2}\Big|_1^2\right) dx$$

$$\left(\frac{4}{2} - \frac{1}{2}\right)$$

$$= \frac{3}{2}\int_1^2 \frac{1}{x}\, dx = \frac{3}{2} \ln|x|\Big|_1^2$$

$$= \frac{3}{2}\left(\ln|2| - \ln|1|\right) = \boxed{\frac{3}{2}\ln 2}.$$